# EQUILIBRIUM STATES AND THEIR ENTROPY DENSITIES IN GAUGE-INVARIANT $C^*$-SYSTEMS

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Abstract. A gauge-invariant $C^*$-system is obtained as the fixed point subalgebra of the infinite tensor product of full matrix algebras under the tensor product unitary action of a compact group. In the paper, thermodynamics is studied in such systems and the chemical potential theory developed by Araki, Haag, Kastler and Takesaki is used. As a generalization of quantum spin system, the equivalence of the KMS condition, the Gibbs condition and the variational principle is shown for translation-invariant states. The entropy density of extremal equilibrium states is also investigated in relation to macroscopic uniformity.

## Introduction

The rigorous treatment of the statistical mechanics of quantum lattice (or spin) systems has been one of the major successes of the $C^*$-algebraic approach to quantum physics. The main results are due to many people but a detailed overview is presented in the monograph [7]. (Chapter 15 of [22] is a concise summary, see also [25].) The usual quantum spin system is described on the infinite tensor product $C^*$-algebra of full matrix algebras. Given an interaction $\Phi$, the local Hamiltonian induces the local dynamics and the local equilibrium state. The global dynamics and the global equilibrium states are obtained by a limiting procedure. The equivalence of the KMS condition, the Gibbs condition and the variational principle for translation-invariant states is the main essence in the theory; they were established around 1970 ([1, 19, 24]). The above mentioned concepts are used to describe equilibrium states. Recently Araki and Moriya extended the ideas to fermionic lattice systems [5].

An attempt to extend quantum statistical mechanics from the setting of spin systems to some approximately finite $C^*$-algebras was made by Kishimoto [17, 18]. Motivated by the chemical potential theory due to Araki, Haag, Kastler and Takesaki [4], in our previous paper [14] we study the equivalence of the KMS condition, the Gibbs condition and the variational principle on approximately finite $C^*$-algebras as a natural extension of the thermodynamics of one-dimensional quantum lattice systems. It turned out that equation (2.8) in the proof of [14, Theorem 2.2] does not hold and the equivalence formulated in that theorem is recovered here under stronger conditions. (The error in the proof was pointed out to the authors by E. Størmer and S. Neshveyev some years ago.)

A gauge-invariant $C^*$-system is obtained as the fixed point subalgebra of the infinite tensor product of full matrix algebras under the tensor product unitary action of a compact group. This situation is a typical example of the chemical potential theory. The primary aim of the present paper is to recover the main results in [14] in the restrictive setup of such gauge-invariant $C^*$-systems. The second aim is to discuss entropy densities and macroscopic

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uniformity for extremal equilibrium states in such $C^*$-systems and to extend the arguments in [13].

## 1. Equilibrium states with chemical potentials

We begin by fixing basic notations and terminologies. Let $M_d(\mathbb{C})$ be the algebra of $d \times d$ complex matrices. Let $\mathcal{F}$ denote a one-dimensional spin (or UHF) $C^*$-algebra $\bigotimes_{k\in\mathbb{Z}} \mathcal{F}_k$ with $\mathcal{F}_k := M_d(\mathbb{C})$, and $\theta$ the right shift on $\mathcal{F}$. Let $G$ be a separable compact group and $\sigma$ a continuous unitary representation of $G$ on $\mathbb{C}^d$ so that a product action $\gamma$ of $G$ on $\mathcal{F}$ is defined by $\gamma_g := \bigotimes_{\mathbb{Z}} \mathrm{Ad}\,\sigma_g$, $g \in G$. Let $\mathcal{A} := \mathcal{F}^\gamma$, the fixed point subalgebra of $\mathcal{F}$ for the action $\gamma$ of $G$. For a finite subset $\Lambda \subset \mathbb{Z}$ let $\mathcal{F}_\Lambda := \bigotimes_{k\in\Lambda} \mathcal{F}_k$ and $\mathcal{A}_\Lambda := \mathcal{A} \cap \mathcal{F}_\Lambda = \mathcal{F}_\Lambda^\gamma$, the fixed point subalgebra for $\gamma|_{\mathcal{F}_\Lambda}$. Then $\mathcal{A}$ is an AF $C^*$-algebra generated by $\{\mathcal{A}_\Lambda\}_{\Lambda\subset\mathbb{Z}}$ ([23, Proposition 2.1]). The algebra $\mathcal{A}$ is called the *observable algebra* while $\mathcal{F}$ is called the *field algebra*. Let $\mathcal{S}(\mathcal{A})$ denote the state space of $\mathcal{A}$ and $\mathcal{S}_\theta(\mathcal{A})$ the set of all $\theta$-invariant states of $\mathcal{A}$.

An *interaction* $\Phi$ is a mapping from the finite subsets of $\mathbb{Z}$ into $\mathcal{A}$ such that $\Phi(\emptyset) = 0$ and $\Phi(X) = \Phi(X)^* \in \mathcal{A}_X$ for each finite $X \subset \mathbb{Z}$. Given an interaction $\Phi$ and a finite subset $\Lambda \subset \mathbb{Z}$, define the *local Hamiltonian* $H_\Lambda$ by

$$H_\Lambda := \sum_{X\subset\Lambda} \Phi(X),$$

and the *surface energy* $W_\Lambda$ by

$$W_\Lambda := \sum\{\Phi(X) : X \cap \Lambda \neq \emptyset,\ X \cap \Lambda^c \neq \emptyset\},$$

whenever the sum converges in norm.

Throughout the paper we assume that an interaction $\Phi$ is *$\theta$-invariant* and has *relatively short range*; namely, $\theta(\Phi(X)) = \Phi(X + 1)$, where $X + 1 := \{k + 1 : k \in X\}$, for every finite $X \subset \mathbb{Z}$ and

$$|||\Phi||| := \sum_{X\ni 0} \frac{\|\Phi(X)\|}{|X|} < \infty,$$

where $|X|$ means the cardinality of $X$. Let $\mathcal{B}(\mathcal{A})$ denote the set of all such interactions, which is a real Banach space with the usual linear operations and the norm $|||\Phi|||$. Moreover, let $\mathcal{B}_0(\mathcal{A})$ denote the set of all $\Phi \in \mathcal{B}(\mathcal{A})$ such that

$$\sum_{X\ni 0} \|\Phi(X)\| < \infty \quad \text{and} \quad \sup_{n\ge 1} \|W_{[1,n]}\| < \infty.$$

Then $\mathcal{B}_0(\mathcal{A})$ is a real Banach space with the norm

$$\|\Phi\|_0 := \sum_{X\ni 0} \|\Phi(X)\| + \sup_{n\ge 1} \|W_{[1,n]}\| \ \ (\ge |||\Phi|||).$$

We define the real Banach space $\mathcal{B}_0(\mathcal{F})$ in a similar manner.

When $\Phi \in \mathcal{B}_0(\mathcal{A})$ we have a strongly continuous one-parameter automorphism group $\alpha^\Phi$ of $\mathcal{F}$ such that

$$\lim_{l,m\to\infty} \|\alpha_t^\Phi(a) - e^{itH_{[-l,m]}} a e^{-itH_{[-l,m]}}\| = 0$$

for all $a \in \mathcal{F}$ uniformly for $t$ in finite intervals (see [15, Theorem 8] and also [7, 6.2.6]). It is straightforward to see that $\alpha_t^\Phi\theta = \theta\alpha_t^\Phi$ and $\alpha_t^\Phi\gamma_g = \gamma_g\alpha_t^\Phi$ for all $t \in \mathbb{R}$ and $g \in G$ so that $\alpha_t^\Phi(\mathcal{A}) = \mathcal{A}$, $t \in \mathbb{R}$. The sextuple $(\mathcal{F}, \mathcal{A}, G, \alpha^\Phi, \gamma, \theta)$ is a so-called *field system* in the chemical potential theory ([4], [7, §5.4.3]). The most general notion of equilibrium states is described by the KMS condition in a general one-parameter $C^*$-dynamical system (see [7, §5.3.1] for example). In this paper we consider only $(\alpha^\Phi, \beta)$-KMS states with $\beta = 1$; so we refer to those states as just $\alpha^\Phi$-*KMS states*. The next proposition says that the $\alpha^\Phi$-KMS states are

automatically $\theta$-invariant. This was stated in [14, Proposition 4.2] but the proof there was given in a wrong way.

**Proposition 1.1.** *Let $\Phi \in \mathcal{B}_0(\mathcal{A})$, and let $K(\mathcal{A},\Phi)$ denote the set of all $\alpha^\Phi$-KMS states of $\mathcal{A}$. Then $K(\mathcal{A},\Phi) \subset \mathcal{S}_\theta(\mathcal{A})$, and $\omega \in K(\mathcal{A},\Phi)$ is extremal in $K(\mathcal{A},\Phi)$ if and only if $\omega$ is extremal in $\mathcal{S}_\theta(\mathcal{A})$.*

*Proof.* The proof below is essentially same as in [10, §III]. Recall that the generator of $\alpha^\Phi$ is the closure of the derivation $\delta_0$ with domain $D(\delta_0) = \bigcup_\Lambda \mathcal{A}_\Lambda$ (over the finite intervals $\Lambda \subset \mathbb{Z}$) given by

$$\delta_0(a) := i \sum_{X\cap\Lambda\neq\emptyset} [\Phi(X), a], \qquad a \in \mathcal{A}_\Lambda.$$

For each $n \in \mathbb{N}$ let $u_n \in \mathcal{F}_{[-n,n]}$ be a unitary implementing the cyclic permutation of $\mathcal{F}_{[-n,n]} = \bigotimes_{-n}^n M_d(\mathbb{C})$, i.e.,

$$\mathrm{Ad}\, u_n(a_{-n} \otimes a_{-n+1} \otimes \cdots \otimes a_{n-1} \otimes a_n) = a_n \otimes a_{-n} \otimes a_{-n+1} \otimes \cdots \otimes a_{n-1}$$

for $a_k \in M_d(\mathbb{C})$. Since $[u_n, \bigotimes_{-n}^n \sigma_g] = 0$, we get $\gamma_g(u_n) = u_n$ for all $g \in G$ so that $u_n \in \mathcal{A}$. Moreover, since $\mathrm{Ad}\, u_n(a) = \theta(a)$ whenever $a \in \mathcal{A}_{[-n,n-1]}$, it is immediate to see that $\theta(a) = \lim_{n\to\infty} \mathrm{Ad}\, u_n(a)$ for all $a \in \mathcal{A}$. Hence, one can apply [10, Corollary II.3] (or [7, 5.3.33A]) to obtain $K(\mathcal{A},\Phi) \subset \mathcal{S}_\theta(\mathcal{A})$, and it suffices to show that $\sup_{n\geq 1} \|\delta_0(u_n)\| < \infty$. This indeed follows because

$$\begin{aligned}
\|\delta_0(u_n)\| &= \Bigg\| \sum_{X\cap[-n,n]\neq\emptyset} [\Phi(X), u_n] \Bigg\| \\
&= \Bigg\| \sum_{X\cap[-n,n]\neq\emptyset} (\Phi(X) - u_n\Phi(X)u_n^*) \Bigg\| \\
&\leq \Bigg\| \sum_{X\subset[-n,n-1]} (\Phi(X) - \theta(\Phi(X))) \Bigg\| + \Bigg\| \sum_{\substack{X\cap[-n,n]\neq\emptyset \\ X\not\subset[-n,n-1]}} (\Phi(X) - u_n\Phi(X)u_n^*) \Bigg\| \\
&\leq \Bigg\| \sum_{X\subset[-n,n-1]} (\Phi(X) - \Phi(X+1)) \Bigg\| + 2\Bigg\| \sum_{\substack{X\cap[-n,n]\neq\emptyset \\ X\not\subset[-n,n-1]}} \Phi(X) \Bigg\| \\
&\leq \sum_{X\ni -n} \|\Phi(X)\| + \sum_{X\ni n} \|\Phi(X)\| + 2\sum_{X\ni n} \|\Phi(X)\| + 2\Bigg\| \sum_{\substack{X\cap[-n,n]\neq\emptyset \\ X\not\subset[-n,n]}} \Phi(X) \Bigg\| \\
&\leq 4\sum_{X\ni 0} \|\Phi(X)\| + 2\|W_{[-n,n]}\| \\
&\leq 4\|\Phi\|_0 < \infty.
\end{aligned}$$

For each $\omega \in \mathcal{S}_\theta(\mathcal{A})$ let $(\pi_\omega, \mathcal{H}_\omega)$ be the GNS cyclic representation of $\mathcal{A}$ associated with $\omega$ and $U_\theta$ be a unitary implementing $\theta$ so that $\pi_\omega(\theta(a)) = U_\theta\pi_\omega(a)U_\theta^*$ for $a \in \mathcal{A}$. Since $(\mathcal{A},\theta)$ is asymptotically abelian in the norm sense, i.e., $\lim_{|n|\to\infty} \|[a, \theta^n(b)]\| = 0$ for all $a, b, \in \mathcal{A}$, it is well known [7, 4.3.14] that

$$\pi_\omega(\mathcal{A})' \cap \{U_\theta\}' \subset \pi_\omega(\mathcal{A})' \cap \pi_\omega(\mathcal{A})''. \tag{1.1}$$

According to [27, Lemma 4.7], the second assertion is a consequence of this together with the first assertion (see also [7, 4.3.17 and 5.3.30 (3)] for extremal points of $\mathcal{S}_\theta(\mathcal{A})$ and of $K(\mathcal{A},\Phi)$). □

**Remark 1.2.** Since $(\mathcal{A},\theta)$ is asymptotically abelian as mentioned in the above proof, $\mathcal{S}_\theta(\mathcal{A})$ becomes a simplex. It is also well known that $K(\mathcal{A},\Phi)$ is a simplex. These were shown in [27, §4], where the lattice (or simplex) structure of state spaces was discussed in a rather general setting. (See also [7, 4.3.11 and 5.3.30 (2)]). Moreover, it is seen from (1.1) [27, Lemma 4.7′] that $K(\mathcal{A},\Phi)$ is a face of $\mathcal{S}_\theta(\mathcal{A})$.

It is known [14, Lemma 4.1] that any tracial state $\phi$ of $\mathcal{A}$ is $\theta$-invariant and $\phi$ is extremal if and only if it is multiplicative in the sense that $\phi(ab)=\phi(a)\phi(b)$ for all $a\in\mathcal{A}_{[i,j]}$ and $b\in\mathcal{A}_{[j+1,k]}$, $i\le j<k$. The $\theta$-invariance of any tracial state of $\mathcal{A}$ is a particular case of Proposition 1.1 where $\Phi$ is identically zero. We denote by $\mathcal{ET}^{\mathrm{f}}(\mathcal{A})$ the set of all faithful and extremal tracial states of $\mathcal{A}$. On the other hand, we denote by $\Xi(G,\sigma)$ the set of all continuous one-parameter subgroups $t\mapsto\xi_t$ of $G$. Two elements $\xi,\xi'$ in $\Xi(G,\sigma)$ are identified if there exists $g\in G$ such that $\mathrm{Ad}\,\sigma_{g^{-1}\xi_t g}=\mathrm{Ad}\,\sigma_{\xi'_t}$, $t\in\mathbb{R}$. In fact, this defines an equivalence relation and we redefine $\Xi(G,\sigma)$ as the set of equivalence classes. Then, [14, Proposition 4.3] says

**Proposition 1.3.** *There is a bijective correspondence $\phi\leftrightarrow\xi$ between $\mathcal{ET}^{\mathrm{f}}(\mathcal{A})$ and $\Xi(G,\sigma)$ under the condition that $\phi$ extends to a $\gamma_\xi$-KMS state of $\mathcal{F}$.*

Let $\tau_0$ be the normalized trace on $M_d(\mathbb{C})$. Let $\phi$ and $\xi$ be as in the above proposition. Then there exists a unique selfadjoint $h\in\mathcal{F}_{\{0\}}=M_d(\mathbb{C})$ such that $\tau_0(e^{-h})=1$ and $\mathrm{Ad}\,\sigma_{\xi_t}=\mathrm{Ad}\,e^{ith}$ for all $t\in\mathbb{R}$. We call this $h$ the *generator* of $\xi$. Note that $\tau_0(e^{-h}\,\cdot)$ is a unique KMS state of $M_d(\mathbb{C})$ with respect to $\mathrm{Ad}\,e^{ith}$ and thus $\hat\phi:=\bigotimes_{\mathbb{Z}}\tau_0(e^{-h}\,\cdot)$ is a unique KMS state of $\mathcal{F}$ with respect to $\gamma_{\xi_t}=\bigotimes_{\mathbb{Z}}\mathrm{Ad}\,e^{ith}$; so $\phi=\hat\phi|_{\mathcal{A}}$.

Let $\Phi\in\mathcal{B}_0(\mathcal{A})$ and $\xi\in\Xi(G,\sigma)$, and let $\omega$ be an $\alpha^\Phi$-KMS state of $\mathcal{A}$. We say that $\xi$ is the *chemical potential* of $\omega$ if there exists an extension $\hat\omega$ of $\omega$ to $\mathcal{F}$ which is a KMS state with respect to $\alpha_t^\Phi\gamma_{\xi_t}$. Let $h$ be the generator of $\xi$, and define a $\theta$-invariant interaction $\Phi^h$ in $\mathcal{F}$ by

$$\Phi^h(X):=\begin{cases}\Phi(\{j\})+\theta^j(h) & \text{if } X=\{j\},\ j\in\mathbb{Z},\\ \Phi(X) & \text{otherwise.}\end{cases} \tag{1.2}$$

Since $\Phi^h\in\mathcal{B}_0(\mathcal{F})$, it generates a one-parameter automorphism group $\alpha^{\Phi^h}$ on $\mathcal{F}$. Then, we have $\alpha_t^{\Phi^h}=\alpha_t^\Phi\gamma_{\xi_t}$, $t\in\mathbb{R}$, and $\alpha^\Phi|_{\mathcal{A}}=\alpha^{\Phi^h}|_{\mathcal{A}}$ ([14, Lemma 4.4]). Due to the uniqueness of an $\alpha^{\Phi^h}$-KMS state of $\mathcal{F}$ ([2, 16]), we notice that there is a unique $\alpha^\Phi$-KMS state with chemical potential $\xi$, which is automatically $\theta$-invariant and faithful. On the other hand, a consequence of the celebrated chemical potential theory in [4, §II] together with Proposition 1.1 is the following: If $\omega$ is a faithful and extremal $\alpha^\Phi$-KMS state of $\mathcal{A}$, then $\omega$ enjoys the chemical potential. A complete conclusion in this direction will be given in Theorem 1.6 below, and Proposition 1.3 is its special case.

To introduce the Gibbs condition, one needs the notion of perturbations of states of $\mathcal{A}$. Let $\omega,\psi\in\mathcal{S}(\mathcal{A})$. For each finite interval $\Lambda\subset\mathbb{Z}$, the *relative entropy* of $\psi_\Lambda:=\psi|_{\mathcal{A}_\Lambda}$ with respect to $\omega_\Lambda:=\omega|_{\mathcal{A}_\Lambda}$ is given by

$$S(\psi_\Lambda,\omega_\Lambda):=\mathrm{Tr}_\Lambda\biggl(\frac{d\psi_\Lambda}{d\mathrm{Tr}_\Lambda}\biggl(\log\frac{d\psi_\Lambda}{d\mathrm{Tr}_\Lambda}-\log\frac{d\omega_\Lambda}{d\mathrm{Tr}_\Lambda}\biggr)\biggr).$$

Here, $\mathrm{Tr}_\Lambda$ denotes the canonical trace on $\mathcal{A}_\Lambda$ such that $\mathrm{Tr}_\Lambda(e)=1$ for any minimal projection $e$ in $\mathcal{A}_\Lambda$. Then the relative entropy $S(\psi,\omega)$ is defined by

$$S(\psi,\omega):=\sup_{\Lambda\subset\mathbb{Z}}S(\psi_\Lambda,\omega_\Lambda)=\lim_{n\to\infty}S(\psi_{[-n,n]},\omega_{[-n,n]}).$$

(See [22] for details on the relative entropy for states of a $C^*$-algebra.) For each $\omega\in\mathcal{S}(\mathcal{A})$ and $Q=Q^*\in\mathcal{A}$, since $\psi\mapsto S(\psi,\omega)+\psi(Q)$ is weakly* lower semicontinuous and strictly convex

on $\mathcal{S}(\mathcal{A})$, the *perturbed state* $[\omega^Q]$ by $Q$ is defined as a unique minimizer of this functional ([8, 22]). Recall [3, 8] that

$$|S(\psi,\omega) - S(\psi,[\omega^Q])| \le 2\|Q\| \tag{1.3}$$

for every $\psi,\omega \in \mathcal{S}(\mathcal{A})$ and $Q = Q^* \in \mathcal{A}$.

Let $\Phi$ be an interaction in $\mathcal{A}$ and $\phi$ a tracial state of $\mathcal{A}$. For each finite $\Lambda \subset \mathbb{Z}$, the *local Gibbs state* $\phi_\Lambda^G$ of $\mathcal{A}_\Lambda$ with respect to $\Phi$ and $\phi$ is defined by

$$\phi_\Lambda^G(a) := \frac{\phi(e^{-H_\Lambda}a)}{\phi(e^{-H_\Lambda})}, \qquad a \in \mathcal{A}_\Lambda.$$

Let $\omega \in \mathcal{S}(\mathcal{A})$ and $(\pi_\omega, \mathcal{H}_\omega, \Omega_\omega)$ be the cyclic representation of $\mathcal{A}$ associated with $\omega$. We say that $\omega$ satisfies the *strong Gibbs condition* if $\Omega_\omega$ is separating for $\pi_\omega(\mathcal{A})''$ and if, for each finite $\Lambda \subset \mathbb{Z}$, there exists a conditional expectation from $\pi_\omega(\mathcal{A})''$ onto $\pi_\omega(\mathcal{A}_\Lambda) \vee \pi_\omega(\mathcal{A}_{\Lambda^c})''$ with respect to $[\omega^{-W_\Lambda}]\tilde{}$ and

$$[\omega^{-W_\Lambda}](ab) = \phi_\Lambda^G(a)[\omega^{-W_\Lambda}](b), \qquad a \in \mathcal{A}_\Lambda,\ b \in \mathcal{A}_{\Lambda^c}. \tag{1.4}$$

Here, $[\omega^{-W_\Lambda}]\tilde{}$ is the normal extension of the perturbed state $[\omega^{-W_\Lambda}]$ to $\pi_\omega(\mathcal{A})''$ (see [14, p. 826]). Furthermore, we say that $\omega$ satisfies the *weak Gibbs condition* with respect to $\Phi$ and $\phi$ if $[\omega^{-W_\Lambda}]|_{\mathcal{A}_\Lambda} = \phi_\Lambda^G$ for any finite $\Lambda \subset \mathbb{Z}$.

Now, let $\Phi \in \mathcal{B}(\mathcal{A})$, $\phi \in \mathcal{ET}^{\mathrm{f}}(\mathcal{A})$ and $\omega \in \mathcal{S}_\theta(\mathcal{A})$. From now on, for simplicity we write $\mathcal{A}_n := \mathcal{A}_{[1,n]}$, $H_n := H_{[1,n]}$, $\phi_n := \phi|_{\mathcal{A}_n}$, $\omega_n := \omega|_{\mathcal{A}_n}$, etc. for each $n \in \mathbb{N}$. The *mean relative entropy* of $\omega$ with respect to $\phi$ is defined by

$$S_{\mathrm{M}}(\omega,\phi) := \lim_{n\to\infty} \frac{1}{n} S(\omega_n,\phi_n) = \sup_{n\ge1} \frac{1}{n} S(\omega_n,\phi_n).$$

(See [14, Lemma 3.1] for justification of the definition.) Define the *mean energy* $A_\Phi$ of $\Phi$ by

$$A_\Phi := \sum_{X \ni 0} \frac{\Phi(X)}{|X|} \quad (\in \mathcal{A}).$$

Furthermore, it is known [14, Theorem 3.5] that $\lim_{n\to\infty} \frac{1}{n}\log\phi(e^{-H_n})$ exists and

$$\lim_{n\to\infty} \frac{1}{n}\log\phi(e^{-H_n}) = \sup\{-S_{\mathrm{M}}(\omega,\phi) - \omega(A_\Phi) : \omega \in \mathcal{S}_\theta(\mathcal{A})\}.$$

The *pressure* of $\Phi$ with respect to $\phi$ is thus defined by

$$p(\Phi,\phi) := \lim_{n\to\infty} \frac{1}{n}\log\phi(e^{-H_n}).$$

We have the variational expressions of $p(\Phi,\phi)$ and $S_{\mathrm{M}}(\omega,\phi)$ as follows.

**Proposition 1.4.** *Let $\phi \in \mathcal{ET}^{\mathrm{f}}(\mathcal{A})$. If $\Phi \in \mathcal{B}(\mathcal{A})$, then*

$$p(\Phi,\phi) = \sup\{-S_{\mathrm{M}}(\omega,\phi) - \omega(A_\Phi) : \omega \in \mathcal{S}_\theta(\mathcal{A})\}. \tag{1.5}$$

*If $\omega \in \mathcal{S}_\theta(\mathcal{A})$, then*

$$-S_{\mathrm{M}}(\omega,\phi) = \inf\{p(\Phi,\phi) + \omega(A_\Phi) : \Phi \in \mathcal{B}(\mathcal{A})\}. \tag{1.6}$$

*Proof.* The expression (1.5) was given in [14, Theorem 3.5] as mentioned above. We further can transform (1.5) into (1.6) by a simple duality argument. In fact, for each $\omega \in \mathcal{S}_\theta(\mathcal{A})$ define $f_\omega \in \mathcal{B}(\mathcal{A})^*$, the dual Banach space of $\mathcal{B}(\mathcal{A})$, by $f_\omega(\Phi) := -\omega(A_\Phi)$, and set $\Gamma := \{f_\omega : \omega \in \mathcal{S}_\theta(\mathcal{A})\}$. Then, it is immediate to see that $\omega \in \mathcal{S}_\theta(\mathcal{A}) \mapsto f_\omega \in \Gamma$ is an affine

homeomorphism in the weak* topologies so that $\Gamma$ is a weakly* compact convex subset of $\mathcal{B}(\mathcal{A})^*$. Define $F : \mathcal{B}(\mathcal{A})^* \to [0, +\infty]$ by

$$\begin{cases} F(f_\omega) := S_{\mathrm{M}}(\omega, \phi) & \text{for } \omega \in \mathcal{S}_\theta(\mathcal{A}), \\ F(g) := +\infty & \text{if } g \in \mathcal{B}(\mathcal{A})^* \setminus \Gamma. \end{cases}$$

Then $F$ is a weakly* lower semicontinuous and convex function on $\mathcal{B}(\mathcal{A})^*$ (see [14, Proposition 3.2]). Since (1.5) means that

$$p(\Phi, \phi) = \sup\{g(\Phi) - F(g) : g \in \mathcal{B}(\mathcal{A})^*\}, \quad \Phi \in \mathcal{B}(\mathcal{A}),$$

it follows by duality (see [9, Proposition I.4.1] for example) that

$$F(g) = \sup\{g(\Phi) - p(\Phi, \phi) : \Phi \in \mathcal{B}(\mathcal{A})\}, \quad g \in \mathcal{B}(\mathcal{A})^*.$$

Hence, for every $\omega \in \mathcal{S}_\theta(\mathcal{A})$,

$$\begin{aligned} S_{\mathrm{M}}(\omega, \phi) &= \sup\{f_\omega(\Phi) - p(\Phi, \phi) : \Phi \in \mathcal{B}(\mathcal{A})\} \\ &= -\inf\{p(\Phi, \phi) + \omega(A_\Phi) : \Phi \in \mathcal{B}(\mathcal{A})\}, \end{aligned}$$

giving (1.6). □

We say that $\omega$ satisfies the *variational principle* with respect to $\Phi$ and $\phi$ if

$$p(\Phi, \phi) = -S_{\mathrm{M}}(\omega, \phi) - \omega(A_\Phi). \tag{1.7}$$

With the above definitions in mind we have the next theorem, recovering main results of [14] (Corollary 3.11 and Theorem 4.5) in the special setup of gauge-invariant $C^*$-systems.

**Theorem 1.5.** *Let $\Phi \in \mathcal{B}_0(\mathcal{A})$, $\phi \in \mathcal{ET}^{\mathrm{f}}(\mathcal{A})$ and $\xi \in \Xi(G, \phi)$ with $\phi \leftrightarrow \xi$ in the sense of Proposition* 1.3. *Then the following conditions for $\omega \in \mathcal{S}(\mathcal{A})$ are equivalent:*

(i) *$\omega$ is an $\alpha^\Phi$-KMS state with chemical potential $\xi$;*
(ii) *$\omega$ satisfies the strong Gibbs condition with respect to $\Phi$ and $\phi$;*
(iii) *$\omega \in \mathcal{S}_\theta(\mathcal{A})$ and $\omega$ satisfies the weak Gibbs condition with respect to $\Phi$ and $\phi$;*
(iv) *$\omega \in \mathcal{S}_\theta(\mathcal{A})$ and $\omega$ satisfies the variational principle with respect to $\Phi$ and $\phi$.*

*Furthermore, there exists a unique $\omega \in \mathcal{S}(\mathcal{A})$ satisfying one* (*hence all*) *of the above conditions.*

*Proof.* (i) ⇒ (ii). Let $\omega$ be an $\alpha^\Phi$-KMS state with chemical potential $\xi$ and $(\pi_\omega, \mathcal{H}_\omega, \Omega_\omega)$ be the associated cyclic representation of $\mathcal{A}$. It is well known that $\Omega_\omega$ is separating for $\pi_\omega(\mathcal{A})''$ (see [7, 5.3.9] for example). According to the proof of [14, Theorem 2.2, (i) ⇒ (ii)], we see that for any finite $\Lambda \subset \mathbb{Z}$ there exists a conditional expectation from $\pi_\omega(\mathcal{A})''$ onto $\pi_\omega(\mathcal{A}_\Lambda) \vee \pi_\omega(\mathcal{A}_{\Lambda^c})''$ with respect to $[\omega^{-W_\Lambda}]^\sim$. (Note that this part of the proof of [14, Theorem 2.2, (i) ⇒ (ii)] is valid.) Moreover, the proof of [14, Theorem 4.5] shows that (1.4) holds for any finite $\Lambda \subset \mathbb{Z}$. Hence we obtain (ii).

(ii) ⇒ (iii). The proof of [14, Theorem 2.2, (ii) ⇒ (i)] guarantees that (ii) implies $\omega \in K(\mathcal{A}, \Phi)$. Hence Proposition 1.1 gives the $\theta$-invariance of $\omega$.

(iii) ⇒ (iv) is contained in [14, Proposition 3.9] proven in a more general setting.

(iv) ⇒ (i). To prove this as well as the last assertion, it suffices to show that a state $\omega \in \mathcal{S}(\mathcal{A})$ satisfying (iv) is unique. First, note that the variational principle (1.7) means that $\Psi \mapsto -\omega(A_\Psi)$ is a tangent functional to the graph of $p(\cdot, \phi)$ on $\mathcal{B}_0(\mathcal{A})$ at $\Phi$. Let $h \in M_d(\mathbb{C})$ be the generator of $\xi$ and $\Phi^h$ a $\theta$-invariant interaction in $\mathcal{F}$ defined by (1.2). Since $\Phi^h \in \mathcal{B}_0(\mathcal{F})$, there is a unique $\alpha^{\Phi^h}$-KMS state $\hat\omega$ of $\mathcal{F}$. Equivalently, there is a unique $\theta$-invariant state $\hat\omega$ of $\mathcal{F}$ satisfying the variational principle with respect to $\Phi^h$, i.e.,

$$P_{\mathcal{F}}(\Phi^h) = s_{\mathcal{F}}(\hat\omega) - \hat\omega(A_{\Phi^h}).$$

Recall here that the pressure $P_{\mathcal{F}}(\Psi)$ of $\Psi \in \mathcal{B}_0(\mathcal{F})$ and the mean entropy $s_{\mathcal{F}}(\psi)$ of $\psi \in \mathcal{S}_\theta(\mathcal{F})$ are

$$P_{\mathcal{F}}(\Psi) := \lim_{n\to\infty} \frac{1}{n} \log \mathrm{Tr}_{\mathcal{F}_n}(e^{-H_n(\Psi)}), \quad s_{\mathcal{F}}(\psi) := \lim_{n\to\infty} \frac{1}{n} S(\psi_n),$$

where $\mathrm{Tr}_{\mathcal{F}_n}$ is the usual trace on $\mathcal{F}_n$ and $H_n(\Psi)$ is the local Hamiltonian of $\Psi$ inside the interval $[1,n]$. The uniqueness property above means (see [9, Proposition I.5.3] for example) that the pressure function $P_{\mathcal{F}}(\cdot)$ on $\mathcal{B}_0(\mathcal{F})$ is differentiable at $\Phi^h$. We have (see [14, (4.11)])

$$p(\Phi,\phi) = P_{\mathcal{F}}(\Phi^h) - \log d, \qquad \Phi \in \mathcal{B}_0(\mathcal{A}). \tag{1.8}$$

By this and (1.2) we obtain

$$p(\Phi+\Psi,\phi) = P_{\mathcal{F}}(\Phi^h+\Psi) - \log d, \qquad \Psi \in \mathcal{B}_0(\mathcal{A}),$$

which implies that $\Psi \in \mathcal{B}_0(\mathcal{A}) \mapsto p(\Psi,\phi)$ is differentiable at $\Phi$. Hence the required implication follows. □

The next theorem is a right formulation of what we wanted to show in [14], though in the restricted setup of gauge-invariant $C^*$-systems.

**Theorem 1.6.** *If $\Phi \in \mathcal{B}_0(\mathcal{A})$ and $\omega \in \mathcal{S}(\mathcal{A})$, then the following conditions are equivalent:*

(i) *$\omega$ is a faithful and extremal $\alpha^\Phi$-KMS state;*
(ii) *$\omega$ is $\alpha^\Phi$-KMS with some chemical potential $\xi \in \Xi(G,\sigma)$;*
(iii) *$\omega$ satisfies the strong Gibbs condition with respect to $\Phi$ and some $\phi \in \mathcal{ET}^{\mathrm{f}}(\mathcal{A})$;*
(iv) *$\omega \in \mathcal{S}_\theta(\mathcal{A})$ and $\omega$ satisfies the weak Gibbs condition with respect to $\Phi$ and some $\phi \in \mathcal{ET}^{\mathrm{f}}(\mathcal{A})$;*
(v) *$\omega \in \mathcal{S}_\theta(\mathcal{A})$ and $\omega$ satisfies the variational principle with respect to $\Phi$ and some $\phi \in \mathcal{ET}^{\mathrm{f}}(\mathcal{A})$.*

*Proof.* In view of Theorem 1.5 we only need to prove the equivalence between (i) and (ii). (i) $\Rightarrow$ (ii) is a consequence of the chemical potential theory in [4, §II] and Proposition 1.1 as mentioned above (after Proposition 1.3). Conversely, suppose (ii) and let $\hat\omega$ be a (unique) KMS state of $\mathcal{F}$ with respect to $\alpha^\Phi\gamma_{\xi_t} = \alpha^{\Phi^h}$ so that $\omega = \hat\omega|_{\mathcal{A}}$. Since $\hat\omega$ is obviously faithful, so is $\omega$. Moreover, the extremality of $\omega$ in $\mathcal{S}_\theta(\mathcal{A})$ follows from that of $\hat\omega$ in $\mathcal{S}_\theta(\mathcal{F})$. This may be well known but we sketch the proof for convenience. Let $(\hat\pi,\hat{\mathcal{H}},\hat\Omega,\hat U_\theta)$ be the cyclic representation of $\mathcal{F}$ associated with $\hat\omega$, where $\hat U_\theta$ is a unitary implementing $\theta$ so that $\hat U_\theta\hat\Omega = \hat\Omega$ and $\hat\pi(\theta(a)) = \hat U_\theta\hat\pi(a)\hat U_\theta^*$ for $a \in \mathcal{F}$. Then the cyclic representation of $\mathcal{A}$ associated with $\omega$ is given by $\mathcal{H}_\omega := \overline{\hat\pi(\mathcal{A})\hat\Omega}$ and $\pi_\omega(a) := \hat\pi(a)|_{\mathcal{H}_\omega}$ for $a \in \mathcal{A}$ with $\Omega_\omega := \hat\Omega$. Let $P : \hat{\mathcal{H}} \to \mathcal{H}_\omega$ be the orthogonal projection. Since $\hat U_\theta P = P\hat U_\theta$, $U_\theta^\omega := \hat U_\theta|_{\mathcal{H}_\omega}$ is a unitary implementing $\theta|_{\mathcal{A}}$. Let $\hat\sigma$ denote the modular automorphism group of $\hat\pi(\mathcal{F})''$ associated with $\hat\Omega$. Since $\hat\sigma_t(\hat\pi(a)) = \hat\pi(\alpha_t^\Phi(a)) \in \hat\pi(\mathcal{A})$ for all $a \in \mathcal{A}$, there exists the conditional expectation $E : \hat\pi(\mathcal{F})'' \to \hat\pi(\mathcal{A})''$ with respect to the state $\langle\,\cdot\,\hat\Omega,\hat\Omega\rangle$ ([26]). Notice that $E$ is $\theta$-covariant, i.e., $E(\hat U_\theta x\hat U_\theta^*) = \hat U_\theta E(x)\hat U_\theta^*$ for all $x \in \hat\pi(\mathcal{F})''$. Now, assume that $\omega_1 \in \mathcal{S}_\theta(\mathcal{A})$ and $\omega_1 \le \lambda\omega$ for some $\lambda > 0$; hence there exists $T_1 \in \pi_\omega(\mathcal{A})'$ with $0 \le T_1 \le \lambda$ such that $\omega_1(a) = \langle T_1\pi_\omega(a)\Omega_\omega,\Omega_\omega\rangle$ for $a \in \mathcal{A}$, and $T_1U_\theta^\omega = U_\theta^\omega T_1$. Define $T := T_1P + (\mathbf{1}-P)$ on $\hat{\mathcal{H}}$. Then it is easy to check that $0 \le T \le \lambda$, $T \in \hat\pi(\mathcal{A})'$ and $T\hat U_\theta = \hat U_\theta T$. Define

$$\hat\omega_1(a) := \langle TE(\hat\pi(a))\hat\Omega,\hat\Omega\rangle, \qquad a \in \mathcal{F},$$

which is a state of $\mathcal{F}$ with $\hat\omega_1|_{\mathcal{A}} = \omega_1$ and $\hat\omega_1 \le \lambda\hat\omega$. For any $a \in \mathcal{F}$ we get

$$\hat\omega_1(\theta(a)) = \langle TE(\hat U_\theta\hat\pi(a)\hat U_\theta^*)\hat\Omega,\hat\Omega\rangle = \langle TE(\hat\pi(a))\hat\Omega,\hat\Omega\rangle = \hat\omega_1(a)$$

so that the extremality of $\hat{\omega}$ implies $\hat{\omega}_1 = \hat{\omega}$ and so $\omega_1 = \omega$. Hence $\omega$ is extremal in $\mathcal{S}_\theta(\mathcal{A})$ (hence in $K(\mathcal{A}, \Phi)$), and (ii) $\Rightarrow$ (i) is shown. $\square$

## 2. More about variational principle

In this section we consider the variational principle for $\omega \in \mathcal{S}_\theta(\mathcal{A})$ in terms of the mean entropy and the pressure which are defined by use of canonical traces on local algebras (not with respect to a tracial state in $\mathcal{ET}^{\mathrm{f}}(\mathcal{A})$). Let $\nu$ be the restriction of $\bigotimes_{\mathbb{Z}} \tau_0$ to $\mathcal{A}$, which is an element of $\mathcal{ET}^{\mathrm{f}}(\mathcal{A})$ corresponding to the trivial chemical potential $\xi = 1$. For each $n \in \mathbb{N}$ the $n$-fold tensor product $\bigotimes_1^n \sigma$ of the unitary representation $\sigma$ is decomposed as

$$\bigotimes_1^n \sigma = m_1\sigma_1 \oplus m_2\sigma_2 \oplus \cdots \oplus m_{K_n}\sigma_{K_n},$$

where $\sigma_i \in \widehat{G}$, $1 \le i \le K_n$, are contained in $\bigotimes_1^n \sigma$ with multiplicities $m_i$. For $1 \le i \le K_n$ let $d_i$ be the dimension of $\sigma_i$. Then, we have $\sum_{i=1}^{K_n} m_i d_i = d^n$ and

$$\mathcal{A}_n = \bigoplus_{i=1}^{K_n} (M_{m_i}(\mathbb{C}) \otimes \mathbf{1}_{d_i}) \cong \bigoplus_{i=1}^{K_n} M_{m_i}(\mathbb{C}), \tag{2.1}$$

$$\mathcal{F}_n \cap \mathcal{A}_n' = \bigoplus_{i=1}^{K_n} (\mathbf{1}_{m_i} \otimes M_{d_i}(\mathbb{C})) \cong \bigoplus_{i=1}^{K_n} M_{d_i}(\mathbb{C}). \tag{2.2}$$

The canonical traces $\mathrm{Tr}_{\mathcal{A}_n}$ on $\mathcal{A}_n$ and $\mathrm{Tr}_{\mathcal{A}_n'}$ on $\mathcal{F}_n \cap \mathcal{A}_n'$ are written as

$$\mathrm{Tr}_{\mathcal{A}_n}\left(\sum_i a_i \otimes \mathbf{1}_{d_i}\right) = \sum_i \mathrm{Tr}_{m_i}(a_i), \qquad a_i \in M_{m_i}(\mathbb{C}),\ 1 \le i \le K_n,$$

$$\mathrm{Tr}_{\mathcal{A}_n'}\left(\sum_i \mathbf{1}_{m_i} \otimes b_i\right) = \sum_i \mathrm{Tr}_{d_i}(b_i), \qquad b_i \in M_{d_i}(\mathbb{C}),\ 1 \le i \le K_n,$$

where $\mathrm{Tr}_m$ denotes the usual trace on $M_m(\mathbb{C})$.

**Lemma 2.1.** (1) *If $\omega \in \mathcal{S}_\theta(\mathcal{A})$, then $\lim_{n\to\infty} \frac{1}{n} S(\omega_n)$ exists and*

$$\lim_{n\to\infty} \frac{1}{n} S(\omega_n) = -S_{\mathrm{M}}(\omega, \nu) + \log d,$$

*where $S(\omega_n)$ is the von Neumann entropy of $\omega_n$ with respect to $\mathrm{Tr}_{\mathcal{A}_n}$, i.e.,*

$$S(\omega_n) := -\mathrm{Tr}_{\mathcal{A}_n}\left(\frac{d\omega_n}{d\mathrm{Tr}_{\mathcal{A}_n}} \log \frac{d\omega_n}{d\mathrm{Tr}_{\mathcal{A}_n}}\right) = -\omega_n\left(\log \frac{d\omega_n}{d\mathrm{Tr}_{\mathcal{A}_n}}\right). \tag{2.3}$$

(2) *If $\Phi \in \mathcal{B}(\mathcal{A})$, then $\lim_{n\to\infty} \frac{1}{n} \log \mathrm{Tr}_{\mathcal{A}_n}(e^{-H_n})$ exists and*

$$\lim_{n\to\infty} \frac{1}{n} \log \mathrm{Tr}_{\mathcal{A}_n}(e^{-H_n}) = p(\Phi, \nu) + \log d.$$

*Proof.* (1) Notice that

$$S(\omega_n) = -S(\omega_n, \nu_n) - \omega_n\left(\log \frac{d\nu_n}{d\mathrm{Tr}_{\mathcal{A}_n}}\right).$$

Representing $\mathcal{A}_n = \bigoplus_{i=1}^{K_n} (M_{m_i}(\mathbb{C}) \otimes \mathbf{1}_{d_i})$ as in (2.1), we have

$$d^n \frac{d\nu_n}{d\mathrm{Tr}_{\mathcal{A}_n}} = \sum_{i=1}^{K_n} d_i \mathbf{1}_{m_i} \otimes \mathbf{1}_{d_i},$$

because

$$d^n\nu_n\biggl(\sum_i a_i\otimes \mathbf{1}_{d_i}\biggr)=\sum_i \mathrm{Tr}_{m_i}(a_i)\mathrm{Tr}_{d_i}(\mathbf{1}_{d_i})=\sum_i d_i\mathrm{Tr}_{m_i}(a_i)$$

for $a_i\in M_{m_i}(\mathbb{C})$, $1\le i\le K_n$. Therefore,

$$\mathbf{1}_{\mathcal{A}_n}\le d^n\frac{d\nu_n}{d\mathrm{Tr}_{\mathcal{A}_n}}\le\Bigl(\max_{1\le i\le K_n}d_i\Bigr)\mathbf{1}_{\mathcal{A}_n}. \tag{2.4}$$

This implies that

$$0\le\omega_n\biggl(\log\frac{d\nu_n}{d\mathrm{Tr}_{\mathcal{A}_n}}\biggr)+n\log d\le\log\Bigl(\max_{1\le i\le K_n}d_i\Bigr).$$

As is well known (see a brief explanation in [14, p. 844] for example), the representation ring of any compact group has polynomial growth; so we have

$$\lim_{n\to\infty}\frac{1}{n}\log\Bigl(\max_{1\le i\le K_n}d_i\Bigr)=0. \tag{2.5}$$

This implies the desired conclusion.

(2) By (2.4) we get

$$\mathrm{Tr}_{\mathcal{A}_n}(e^{-H_n})\le d^n\nu_n(e^{-H_n})\le\Bigl(\max_{1\le i\le K_n}d_i\Bigr)\mathrm{Tr}_{\mathcal{A}_n}(e^{-H_n}),$$

implying the result. □

In view of the above lemma we define the *mean entropy* of $\omega\in\mathcal{S}_\theta(\mathcal{A})$ by

$$s_{\mathcal{A}}(\omega):=\lim_{n\to\infty}\frac{1}{n}S(\omega_n)\quad(=-S_{\mathrm{M}}(\omega,\nu)+\log d),$$

and the *pressure* of $\Phi\in\mathcal{B}(\mathcal{A})$ by

$$P_{\mathcal{A}}(\Phi):=\lim_{n\to\infty}\frac{1}{n}\log\mathrm{Tr}_{\mathcal{A}_n}(e^{-H_n})\quad(=p(\Phi,\nu)+\log d).$$

The variational expression (1.5) in case of $\phi=\nu$ is rewritten as

$$P_{\mathcal{A}}(\Phi)=\sup\{s_{\mathcal{A}}(\omega)-\omega(A_\Phi):\omega\in\mathcal{S}_\theta(\mathcal{A})\}.$$

**Proposition 2.2.** *Let $\Phi\in\mathcal{B}_0(\mathcal{A})$ and $\xi\in\Xi(G,\sigma)$ with the generator $h$. Assume that $\xi$ is central, i.e., $\xi_t$ belongs to the center of $G$ for any $t$ (this is the case if $G$ is abelian). Then $\Phi^h$ defined by (1.2) is an interaction in $\mathcal{A}$, and $\omega\in\mathcal{S}_\theta(\mathcal{A})$ is $\alpha^\Phi$-KMS with chemical potential $\xi$ if and only if it satisfies the variational principle*

$$P_{\mathcal{A}}(\Phi^h)=s_{\mathcal{A}}(\omega)-\omega(A_{\Phi^h}). \tag{2.6}$$

*In particular, $\omega$ is $\alpha^\Phi$-KMS with trivial chemical potential if and only if it satisfies*

$$P_{\mathcal{A}}(\Phi)=s_{\mathcal{A}}(\omega)-\omega(A_\Phi).$$

*Proof.* The assumption of $\xi$ being central implies that $\mathrm{Ad}\,\sigma_g(\sigma_{\xi_t})=\sigma_{\xi_t}$ for all $g\in G$ and $t\in\mathbb{R}$. Hence, it is immediate to see that $\bigotimes_\Lambda e^{-h}=\exp\bigl(-\sum_{j\in\Lambda}\theta^j(h)\bigr)$ is in $\mathcal{A}_\Lambda$ for any finite $\Lambda\subset\mathbb{Z}$ and so the interaction $\Phi^h$ is in $\mathcal{A}$. Let $\phi$ be an element of $\mathcal{ET}^{\mathrm{f}}(\mathcal{A})$ corresponding to $\xi$ as in Proposition 1.3. We may show that (2.6) is equivalent to the variational principle (1.7) with respect to $\phi$. Since $A_{\Phi^h}=A_\Phi+h$, it suffices to prove the following two expressions:

$$p(\Phi,\phi)=P_{\mathcal{A}}(\Phi^h)-\log d \tag{2.7}$$

and for every $\omega\in\mathcal{S}_\theta(\mathcal{A})$

$$-S_{\mathrm{M}}(\omega,\phi)=s_{\mathcal{A}}(\omega)-\omega(h)-\log d. \tag{2.8}$$

Let $H_n(\Phi^h)$ be the local Hamiltonian of $\Phi^h$ inside the interval $[1,n]$. Since

$$\phi_n(e^{-H_n}) = \nu_n\left(\left(\bigotimes_{j=1}^n e^{-h}\right)e^{-H_n}\right) = \nu_n(e^{-H_n(\Phi^h)}) = \mathrm{Tr}_{\mathcal{A}_n}\left(\frac{d\nu_n}{d\mathrm{Tr}_{\mathcal{A}_n}} e^{-H_n(\Phi^h)}\right),$$

we obtain (2.7) thanks to (2.4) and (2.5). On the other hand, since

$$\begin{aligned} -S(\omega_n,\phi_n) &= S(\omega_n) + \omega_n\left(\log\frac{d\phi_n}{d\mathrm{Tr}_{\mathcal{A}_n}}\right) \\ &= S(\omega_n) + \omega_n\left(\log\left(\frac{d\nu_n}{d\mathrm{Tr}_{\mathcal{A}_n}} \bigotimes_{j=1}^n e^{-h}\right)\right) \\ &= S(\omega_n) - n\omega(h) + \omega_n\left(\log\frac{d\nu_n}{d\mathrm{Tr}_{\mathcal{A}_n}}\right), \end{aligned} \tag{2.9}$$

the expression (2.8) follows. □

## 3. Entropy densities

From now on let $\mathcal{F}$, $G$, $\sigma$, $\gamma$, $\mathcal{A}$, $\theta$, etc. be as in the previous sections. Let $\Phi \in \mathcal{B}_0(\mathcal{A})$ be given and $\alpha^\Phi$ be the associated one-parameter automorphism group. Furthermore, let $\phi \in \mathcal{ET}^{\mathrm{f}}(\mathcal{A})$ and the corresponding $\xi \in \Xi(G,\sigma)$ with generator $h$ be given as in Proposition 1.3; hence $\phi$ extends to the $\gamma_\xi$-KMS state $\hat\phi$ of $\mathcal{F}$. For each $n \in \mathbb{N}$ we then have the local Gibbs state of $\mathcal{A}_n$ with respect to $\Phi$ and $\phi$ given by

$$\phi_n^G(a) := \frac{\phi(e^{-H_n}a)}{\phi(e^{-H_n})}, \qquad a \in \mathcal{A}_n,$$

and the local Gibbs state of $\mathcal{F}_n$ with respect to $\Phi^h$ given by

$$\hat\phi_n^G(a) := \frac{\mathrm{Tr}_{\mathcal{F}_n}(e^{-H_n(\Phi^h)}a)}{\mathrm{Tr}_{\mathcal{F}_n}(e^{-H_n(\Phi^h)})}. \qquad a \in \mathcal{F}_n.$$

The notation $\hat\phi_n^G$ is justified as follows: Since $\bigotimes_1^n e^{-h}$ and $e^{-H_n}$ commute (see the proof of [14, Proposition 4.3]), $\hat\phi_n^G$ is written as

$$\hat\phi_n^G(a) = \frac{\mathrm{Tr}_{\mathcal{F}_n}\big((\bigotimes_1^n e^{-h})e^{-H_n}a\big)}{\mathrm{Tr}_{\mathcal{F}_n}\big((\bigotimes_1^n e^{-h})e^{-H_n}\big)} = \frac{\hat\phi(e^{-H_n}a)}{\hat\phi(e^{-H_n})}, \qquad a \in \mathcal{F}_n. \tag{3.1}$$

With these notations we have

**Theorem 3.1.** *Let $\omega$ be an $\alpha^\Phi$-KMS state of $\mathcal{A}$ with chemical potential $\xi$ and $\hat\omega$ be the $\alpha^\Phi\gamma_\xi$-KMS state of $\mathcal{F}$ extending $\omega$. Then*

$$\begin{aligned} S_{\mathrm{M}}(\omega,\phi) &= \lim_{n\to\infty}\frac{1}{n}S(\phi_n^G,\phi_n) = \lim_{n\to\infty}\frac{1}{n}S(\hat\phi_n^G,\hat\phi_n) \\ &= S_{\mathrm{M}}(\hat\omega,\hat\phi) = -s_{\mathcal{F}}(\hat\omega) + \hat\omega(h) + \log d \end{aligned}$$

*and*

$$s_{\mathcal{F}}(\hat\omega) = \lim_{n\to\infty}\frac{1}{n}S(\hat\phi_n^G) = \lim_{n\to\infty}\frac{1}{n}S(\phi_n^G),$$

*where $s_{\mathcal{F}}(\hat\omega) := \lim_{n\to\infty}\frac{1}{n}S(\hat\omega_n)$, the mean entropy of $\hat\omega$. In particular, if $\xi$ is central, then $s_{\mathcal{A}}(\omega) = s_{\mathcal{F}}(\hat\omega)$.*

*Proof.* The following proof of $S_{\mathrm{M}}(\omega,\phi)=\lim_{n\to\infty}\frac{1}{n}S(\phi_n^G,\phi_n)$ is a slight modification of [20, Theorem 2.1]. The proof of Theorem 1.5 says that $\Psi\in\mathcal{B}_0(\mathcal{A})\mapsto p(\Psi,\phi)$ is differentiable at $\Phi$ with the tangent functional $\Psi\in\mathcal{B}_0(\mathcal{A})\mapsto-\omega(A_\Psi)$. Hence we have

$$\frac{d}{d\beta}\bigg|_{\beta=1}p(\beta\Phi,\phi)=-\omega(A_\Phi). \tag{3.2}$$

Furthermore, we obtain

$$\frac{d}{d\beta}\bigg|_{\beta=1}\frac{1}{n}\log\phi(e^{-H_n(\beta\Phi)})=\frac{1}{n}\,\frac{\phi(e^{-H_n}(-H_n))}{\phi(e^{-H_n})}=-\frac{1}{n}\phi_n^G(H_n), \tag{3.3}$$

and as in [20]

$$\lim_{n\to\infty}\frac{d}{d\beta}\bigg|_{\beta=1}\frac{1}{n}\log\phi(e^{-H_n(\beta\Phi)})=\frac{d}{d\beta}\bigg|_{\beta=1}p(\beta\Phi,\phi). \tag{3.4}$$

Combining (3.2)–(3.4) yields $\lim_{n\to\infty}\frac{1}{n}\phi_n^G(H_n)=\omega(A_\Phi)$. Therefore, Theorem 1.5 implies

$$\begin{aligned}
S_{\mathrm{M}}(\omega,\phi)&=-p(\Phi,\phi)-\omega(A_\Phi)\\
&=\lim_{n\to\infty}\frac{1}{n}\big(-\log\phi(e^{-H_n})-\phi_n^G(H_n)\big)\\
&=\lim_{n\to\infty}\frac{1}{n}\phi_n^G\left(\log\frac{d\phi_n^G}{d\phi_n}\right)\\
&=\lim_{n\to\infty}\frac{1}{n}S(\phi_n^G,\phi_n).
\end{aligned}$$

On the other hand, $\hat\omega$ satisfies the variational principle with respect to $\Phi^h$, i.e.,

$$P_{\mathcal{F}}(\Phi^h)=s_{\mathcal{F}}(\hat\omega)-\hat\omega(A_{\Phi^h}).$$

Since $A_{\Phi^h}=A_\Phi+h$, this and (1.8) imply

$$\begin{aligned}
S_{\mathrm{M}}(\omega,\phi)&=-p(\Phi,\phi)-\omega(A_\Phi)\\
&=-s_{\mathcal{F}}(\hat\omega)+\hat\omega(A_\Phi+h)+\log d-\omega(A_\Phi)\\
&=-s_{\mathcal{F}}(\hat\omega)+\hat\omega(h)+\log d.
\end{aligned} \tag{3.5}$$

Since $d\hat\phi_n/d\mathrm{Tr}_{\mathcal{F}_n}=d^{-n}\bigotimes_{j=1}^n e^{-h}$, we have

$$\begin{aligned}
S(\hat\omega_n,\hat\phi_n)&=-S(\hat\omega_n)-\hat\omega_n\left(\log\frac{d\hat\phi_n}{d\mathrm{Tr}_{\mathcal{F}_n}}\right)\\
&=-S(\hat\omega_n)+\hat\omega\left(\sum_{j=1}^n\theta^j(h)\right)+n\log d\\
&=-S(\hat\omega_n)+n\hat\omega(h)+n\log d
\end{aligned}$$

so that

$$S_{\mathrm{M}}(\hat\omega,\hat\phi)=-s_{\mathcal{F}}(\hat\omega)+\hat\omega(h)+\log d.$$

Furthermore,

$$\begin{aligned}
S(\hat\phi_n^G,\hat\phi_n)&=-S(\hat\phi_n^G)+\sum_{j=1}^n\hat\phi_n^G(\theta^j(h))+n\log d\\
&=-S(\hat\phi_n^G)+\sum_{j=1}^n\hat\phi_{[1-j,\,n-j]}^G(h)+n\log d.
\end{aligned}$$

By [20] we have $s_{\mathcal{F}}(\hat\omega) = \lim_{n\to\infty} \frac{1}{n} S(\hat\phi_n^G)$. The uniqueness of $\alpha^\Phi\gamma_\xi$ $(= \alpha^{\Phi^h})$-KMS state implies that $\hat\phi^G_{[-\ell,m]} \to \hat\omega$ weakly* as $\ell, m \to \infty$. For each $\varepsilon > 0$ one can choose $n_0 \in \mathbb{N}$ such that $\bigl|\hat\phi^G_{[-\ell,\,m]}(h) - \hat\omega(h)\bigr| \le \varepsilon$ for all $\ell, m \ge n_0$. If $n > 2n_0$ and $n_0 < j \le n - n_0$, then $j - 1 \ge n_0$ and $n - j \ge n_0$ so that $\bigl|\hat\phi^G_{[1-j,\,n-j]}(h) - \hat\omega(h)\bigr| \le \varepsilon$. Hence we have

$$\left| \frac{1}{n} \sum_{j=1}^{n} \hat\phi^G_{[1-j,\,n-j]}(h) - \hat\omega(h) \right| \le \frac{4\|h\| n_0}{n} + \varepsilon.$$

This shows that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \hat\phi^G_{[1-j,\,n-j]}(h) = \hat\omega(h).$$

Therefore,

$$\lim_{n\to\infty} \frac{1}{n} S(\hat\phi_n^G, \hat\phi_n) = -s_{\mathcal{F}}(\hat\omega) + \hat\omega(h) + \log d,$$

and the proof of the first part is completed.

The last assertion follows from (2.6) and (3.5). It remains to prove

$$\lim_{n\to\infty} \frac{1}{n} S(\phi_n^G) = \lim_{n\to\infty} \frac{1}{n} S(\hat\phi_n^G). \tag{3.6}$$

To prove this we give a lemma.

**Lemma 3.2.** *Under* (2.1) *and* (2.2) *let*

$$D^0 = \sum_{i=1}^{K_n} D_i^0 \otimes \mathbf{1}_{d_i} \in \mathcal{A}_n, \quad D' = \sum_{i=1}^{K_n} \mathbf{1}_{m_i} \otimes D_i' \in \mathcal{F}_n \cap \mathcal{A}_n'$$

*with positive semidefinite matrices $D_i^0 \in M_{m_i}(\mathbb{C})$ and $D_i' \in M_{d_i}(\mathbb{C})$ such that $\mathrm{Tr}_{\mathcal{F}_n}(D^0 D') = 1$. Then $D := D^0 D'$ is a density matrix with respect to $\mathrm{Tr}_{\mathcal{F}_n}$. If $D|_{\mathcal{A}_n}$ is the density matrix of $\mathrm{Tr}_{\mathcal{F}_n}(D\,\cdot)|_{\mathcal{A}_n}$ with respect to $\mathrm{Tr}_{\mathcal{A}_n}$, then*

$$|S(D|_{\mathcal{A}_n}) - S(D)| \le \log\left( \max_{1\le i\le K_n} d_i \right),$$

*where $S(D)$ is the von Neumann entropy of $D$ with respect to $\mathrm{Tr}_{\mathcal{F}_n}$ and $S(D|_{\mathcal{A}_n})$ is that of $D|_{\mathcal{A}_n}$ with respect to $\mathrm{Tr}_{\mathcal{A}_n}$ (see* (2.3)*).*

*Proof.* The first assertion is obvious. Let $E_{\mathcal{A}_n}$ denote the conditional expectation from $\mathcal{F}_n$ onto $\mathcal{A}_n$ with respect to $\mathrm{Tr}_{\mathcal{F}_n}$. Notice that

$$\begin{aligned} S(E_{\mathcal{A}_n}(D)) - S(D) &= \mathrm{Tr}_{\mathcal{F}_n}(D \log D - E_{\mathcal{A}_n}(D) \log E_{\mathcal{A}_n}(D)) \\ &= S(D, E_{\mathcal{A}_n}(D)), \end{aligned}$$

the relative entropy of the densities $D$ and $E_{\mathcal{A}_n}(D)$ in $\mathcal{F}_n$. Set $H_i^0 := D_i^0/\mathrm{Tr}_{m_i}(D_i^0)$, $H_i' := D_i'/\mathrm{Tr}_{d_i}(D_i')$ and $D_i := H_i^0 \otimes H_i'$. The joint convexity of relative entropy implies

$$S(D, E_{\mathcal{A}_n}(D)) \le \sum_{i=1}^{K_n} \mathrm{Tr}_{m_i}(D_i^0)\mathrm{Tr}_{d_i}(D_i') S(D_i, E_{\mathcal{A}_n}(D_i)).$$

Since $E_{\mathcal{A}_n}(D_i) = d_i^{-1} H_i^0 \otimes \mathbf{1}_{d_i}$, we get

$$\begin{aligned} &S(D_i, E_{\mathcal{A}_n}(D_i)) \\ &\quad = \mathrm{Tr}_{\mathcal{F}_n}\Big(D_i\Big(\log H_i^0 \otimes \mathbf{1}_{d_i} + \mathbf{1}_{m_i} \otimes \log H_i' - \log H_i^0 \otimes \mathbf{1}_{d_i} + (\log d_i)\mathbf{1}_{m_i} \otimes \mathbf{1}_{d_i}\Big)\Big) \\ &\quad = \mathrm{Tr}_{d_i}(H_i' \log H_i') + \log d_i \\ &\quad \le \log d_i. \end{aligned}$$

Therefore,

$$0 \le S(E_{\mathcal{A}_n}(D)) - S(D) \le \log\Big(\max_{1\le i\le K_n} d_i\Big). \tag{3.7}$$

Next, since for $a = \sum_i a_i \otimes \mathbf{1}_{d_i} \in \mathcal{A}_n$

$$\begin{aligned} \mathrm{Tr}_{\mathcal{F}_n}(aD') &= \mathrm{Tr}_{\mathcal{F}_n}\left(\sum_{i=1}^{K_n} a_i \otimes D_i'\right) = \sum_{i=1}^{K_n} \mathrm{Tr}_{m_i}(a_i)\mathrm{Tr}_{d_i}(D_i') \\ &= \mathrm{Tr}_{\mathcal{F}_n}\left(\left(\sum_{i=1}^{K_n} \frac{\mathrm{Tr}_{d_i}(D_i')}{d_i}\mathbf{1}_{m_i} \otimes \mathbf{1}_{d_i}\right)a\right), \end{aligned}$$

we get

$$E_{\mathcal{A}_n}(D') = \sum_{i=1}^{K_n} \frac{\mathrm{Tr}_{d_i}(D_i')}{d_i}\mathbf{1}_{m_i} \otimes \mathbf{1}_{d_i}$$

so that

$$E_{\mathcal{A}_n}(D) = D^0 E_{\mathcal{A}_n}(D') = \sum_{i=1}^{K_n} \frac{\mathrm{Tr}_{d_i}(D_i')}{d_i} D_i^0 \otimes \mathbf{1}_{d_i}.$$

Hence we have

$$\begin{aligned} &S(E_{\mathcal{A}_n}(D)) \\ &\quad = -\mathrm{Tr}_{\mathcal{F}_n}\left(\sum_{i=1}^{K_n} \frac{\mathrm{Tr}_{d_i}(D_i')}{d_i} D_i^0 \otimes \mathbf{1}_{d_i}\Big(\log D_i^0 \otimes \mathbf{1}_{d_i} + \Big(\log \mathrm{Tr}_{d_i}(D_i') - \log d_i\Big)\mathbf{1}_{m_i} \otimes \mathbf{1}_{d_i}\right) \\ &\quad = -\sum_{i=1}^{K_n} \mathrm{Tr}_{d_i}(D_i')\mathrm{Tr}_{m_i}(D_i^0 \log D_i^0) - \sum_{i=1}^{K_n} \mathrm{Tr}_{m_i}(D_i^0)\mathrm{Tr}_{d_i}(D_i')\Big(\log \mathrm{Tr}_{d_i}(D_i') - \log d_i\Big). \end{aligned}$$

On the other hand, since $D|_{\mathcal{A}_n}$ is $\sum_{i=1}^{K_n} \mathrm{Tr}_{d_i}(D_i')D_i^0$ as an element of $\bigoplus_{i=1}^{K_n} M_{m_i}(\mathbb{C})$, we have

$$\begin{aligned} S(D|_{\mathcal{A}_n}) &= -\sum_{i=1}^{K_n} \mathrm{Tr}_{m_i}\Big(\mathrm{Tr}_{d_i}(D_i')D_i^0\Big(\log D_i^0 + \log \mathrm{Tr}_{d_i}(D_i')\Big)\Big) \\ &= -\sum_{i=1}^{K_n} \mathrm{Tr}_{d_i}(D_i')\mathrm{Tr}_{m_i}(D_i^0 \log D_i^0) - \sum_{i=1}^{K_n} \mathrm{Tr}_{m_i}(D_i^0)\mathrm{Tr}_{d_i}(D_i') \log \mathrm{Tr}_{d_i}(D_i'). \end{aligned}$$

Therefore,

$$S(E_{\mathcal{A}_n}(D)) - S(D|_{\mathcal{A}_n}) = \sum_{i=1}^{K_n} \mathrm{Tr}_{m_i}(D_i^0)\mathrm{Tr}_{d_i}(D_i') \log d_i$$

so that

$$0 \le S(E_{\mathcal{A}_n}(D)) - S(D|_{\mathcal{A}_n}) \le \log\Big(\max_{1\le i\le K_n} d_i\Big). \tag{3.8}$$

Combining (3.7) and (3.8) gives the conclusion. $\square$

*Proof of* (3.6). Let $\hat{D}_n^G$ be the density of the local Gibbs state $\hat{\phi}_n^G$ with respect to $\mathrm{Tr}_{\mathcal{F}_n}$, which is written as

$$\hat{D}_n^G = \frac{(\bigotimes_1^n e^{-h})e^{-H_n}}{\mathrm{Tr}_{\mathcal{F}_n}\big((\bigotimes_1^n e^{-h})e^{-H_n}\big)}. \tag{3.9}$$

This is obviously of the form of $D$ in Lemma 3.2, i.e., the product of an element of $\mathcal{A}_n$ and an element of $\mathcal{F}_n \cap \mathcal{A}_n'$. Furthermore, since $\mathrm{Tr}_{\mathcal{F}_n}(\hat{D}_n^G\,\cdot)|_{\mathcal{A}_n} = \hat{\phi}_n^G|_{\mathcal{A}_n} = \phi_n^G$ thanks to (3.1), it follows that the density of $\phi_n^G$ with respect to $\mathrm{Tr}_{\mathcal{A}_n}$ is $\hat{D}_n^G|_{\mathcal{A}_n}$ (in the notation of Lemma 3.2). Hence, Lemma 3.2 implies

$$\big|S(\phi_n^G) - S(\hat{\phi}_n^G)\big| \le \log\bigg(\max_{1\le i\le K_n} d_i\bigg)$$

so that we obtain (3.6) thanks to (2.5). $\square$

## 4. Macroscopic uniformity

Let $\phi \in \mathcal{ET}^{\mathrm{f}}(\mathcal{A})$ and $0 < \varepsilon < 1$. For each $n \in \mathbb{N}$ and for each state $\psi$ of $\mathcal{A}_n$ we define the two quantities

$$\beta_\varepsilon(\psi) := \min\{\mathrm{Tr}_{\mathcal{A}_n}(q) : q \in \mathcal{A}_n \text{ is a projection with } \psi(q) \ge 1-\varepsilon\},$$
$$\beta_\varepsilon(\psi, \phi_n) := \min\{\phi_n(q) : q \in \mathcal{A}_n \text{ is a projection with } \psi(q) \ge 1-\varepsilon\}.$$

For each state $\psi'$ of $\mathcal{F}_n$ the quantities $\beta_\varepsilon(\psi')$ and $\beta_\varepsilon(\psi', \hat{\phi}_n)$ are defined in a similar way with $\mathcal{F}_n$ instead of $\mathcal{A}_n$. The aim of this section is to prove

**Theorem 4.1.** *Let $\Phi$, $\phi$, $\xi$ and $h$ be as in Theorem* 1.5, *and let $\omega$ be an $\alpha^\Phi$-KMS state of $\mathcal{A}$ with chemical potential $\xi$. Then, for every* $0 < \varepsilon < 1$,

$$-S_{\mathrm{M}}(\omega, \phi) = \lim_{n\to\infty} \frac{1}{n} \log \beta_\varepsilon(\omega_n, \phi_n) \tag{4.1}$$
$$= \lim_{n\to\infty} \frac{1}{n} \log \beta_\varepsilon(\phi_n^G, \phi_n) = \lim_{n\to\infty} \frac{1}{n} \log \beta_\varepsilon(\hat{\phi}_n^G, \hat{\phi}_n). \tag{4.2}$$

*Moreover, if $\xi$ is central, then for every* $0 < \varepsilon < 1$,

$$s_{\mathcal{A}}(\omega) = \lim_{n\to\infty} \frac{1}{n} \log \beta_\varepsilon(\omega_n) = \lim_{n\to\infty} \frac{1}{n} \log \beta_\varepsilon(\phi_n^G) \tag{4.3}$$
$$= \lim_{n\to\infty} \frac{1}{n} \log \beta_\varepsilon(\hat{\omega}_n) = \lim_{n\to\infty} \frac{1}{n} \log \beta_\varepsilon(\hat{\phi}_n^G). \tag{4.4}$$

To prove the theorem, we modify the proofs of [13, Theorems 3.1 and 3.3]. Let $\omega$ be as in the theorem and $(\pi_\omega, \mathcal{H}_\omega, \Omega_\omega)$ be the cyclic representation of $\mathcal{A}$ associated with $\omega$. For each $n \in \mathbb{N}$ set

$$D_n := \frac{d\omega_n}{d\phi_n} \quad \text{and} \quad D_n^G := \frac{d\phi_n^G}{d\phi_n} = \frac{e^{-H_n}}{\phi(e^{-H_n})}.$$

**Lemma 4.2.** *For every $n \in \mathbb{N}$,*

$$\log D_n^G - \log D_n \le 2\|W_n\|.$$

*Proof.* For every state $\psi$ of $\mathcal{A}_n$ let $\tilde{\psi}$ be the state of $\pi_\omega(\mathcal{A}_n)$ such that $\psi = \tilde{\psi} \circ \pi_\omega|_{\mathcal{A}_n}$; in particular, let $\tilde{\phi}_n^G$ be that for $\phi_n^G$. Moreover, let $\tilde{\omega}$ be the normal extension of $\omega$ to $\pi_\omega(\mathcal{A})''$; so $\tilde{\omega}_n = \tilde{\omega}|_{\pi_\omega(\mathcal{A}_n)}$. Note (see [14, p. 826]) that the normal extension $[\omega^{-W_n}]^\sim$ of $[\omega^{-W_n}]$ coincides with the perturbed state $[\tilde{\omega}^{-\pi_\omega(W_n)}]$. There exists the conditional expectation $E_n$ from $\pi_\omega(\mathcal{A})''$ onto $\pi_\omega(\mathcal{A}_n)$ with respect to $[\omega^{-W_n}]^\sim$ because $\pi_\omega(\mathcal{A}_n)$ is globally invariant under

the modular automorphism associated with this state. (See the proof of [14, Theorem 2.2, (i) $\Rightarrow$ (ii)]; this part of the proof of [14, Theorem 2.2] is valid.) Then, we successively estimate

$$\begin{aligned} S(\psi,\omega_n) &= S(\tilde\psi,\tilde\omega_n) \le S(\tilde\psi\circ E_n,\tilde\omega) \\ &\le S(\tilde\psi\circ E_n,[\omega^{-W_n}]\tilde{}\,) + 2\|W_n\| \\ &= S(\tilde\psi\circ E_n,\tilde\phi_n^G\circ E_n) + 2\|W_n\| \\ &= S(\psi,\phi_n^G) + 2\|W_n\|. \end{aligned} \tag{4.5}$$

Here, the first inequality is the monotonicity of relative entropy ([22, 5.12 (iii)]) under the restriction of the states of $\pi_\omega(\mathcal{A})''$ to its subalgebra $\pi_\omega(\mathcal{A}_n)$, and the second is due to (1.3). The second equality follows because Theorem 1.5 ((ii) or (iii)) gives $[\omega^{-W_n}]\tilde{}=\tilde\phi_n^G\circ E_n$. The last equality is seen by applying the monotonicity of relative entropy in two ways (or by [22, 5.15]). We now obtain

$$\psi(\log D_n^G-\log D_n)=S(\psi,\omega_n)-S(\psi,\phi_n^G)\le 2\|W_n\|$$

for all states $\psi$ of $\mathcal{A}_n$, which implies the conclusion. ☐

**Lemma 4.3.** *For the densities $D_n$ and $D_n^G$,*

$$\lim_{n\to\infty}\frac{1}{n}\pi_\omega(-\log D_n)=\lim_{n\to\infty}\frac{1}{n}\pi_\omega(-\log D_n^G)=-S_{\mathrm{M}}(\omega,\phi)\mathbf{1}\quad\textit{strongly.}$$

*Proof.* Since $\omega$ is extremal in $\mathcal{S}_\theta(\mathcal{A})$, the mean ergodic theorem says that

$$\lim_{n\to\infty}\frac{1}{n}\pi_\omega\left(\sum_{j=1}^n\theta^j(A_\Phi)\right)=\omega(A_\Phi)\mathbf{1}\quad\text{strongly.}$$

Since it follows as in [13] that

$$\lim_{n\to\infty}\frac{1}{n}\left\|\sum_{j=1}^n\theta^j(A_\Phi)-H_n\right\|=0,$$

we have

$$\lim_{n\to\infty}\frac{1}{n}\pi_\omega(H_n)=\omega(A_\Phi)\mathbf{1}\quad\text{strongly.} \tag{4.6}$$

Therefore, we obtain the strong convergence

$$\begin{aligned} \frac{1}{n}\pi_\omega(-\log D_n^G) &= \frac{1}{n}\pi_\omega(H_n)+\frac{1}{n}\big(\log\phi(e^{-H_n})\big)\mathbf{1} \\ &\longrightarrow \big(\omega(A_\Phi)+p(\Phi,\phi)\big)\mathbf{1}=-S_{\mathrm{M}}(\omega,\phi)\mathbf{1} \end{aligned} \tag{4.7}$$

due to the variational principle of $\omega$ in Theorem 1.5.

Next, let $a_n:=-\frac{1}{n}\log D_n$ and $b_n:=-\frac{1}{n}\log D_n^G+\frac{2}{n}\|W_n\|$; so $\pi_\omega(b_n)\to -S_{\mathrm{M}}(\omega,\phi)\mathbf{1}$ strongly by what is already shown. We get $a_n\le b_n$ by Lemma 4.2, and moreover

$$\begin{aligned} a_n &= -\frac{1}{n}\log\frac{d\omega_n}{d\mathrm{Tr}_{\mathcal{A}_n}}+\frac{1}{n}\log\frac{d\phi_n}{d\mathrm{Tr}_{\mathcal{A}_n}}\ge\frac{1}{n}\log\frac{d\phi_n}{d\mathrm{Tr}_{\mathcal{A}_n}} \\ &= -\frac{1}{n}\sum_{j=1}^n\theta^j(h)+\frac{1}{n}\log\frac{d\nu_n}{d\mathrm{Tr}_{\mathcal{A}_n}}\ge-\|h\|-\log d \end{aligned}$$

(see (2.9) and (2.4)). Hence $\{b_n-a_n\}$ is uniformly bounded. Since

$$\begin{aligned} \|\pi_\omega(b_n-a_n)\Omega_\omega\|^2 &\le \Big(\sup_m\|b_m-a_m\|\Big)\omega(b_n-a_n) \\ &\longrightarrow \Big(\sup_m\|b_m-a_m\|\Big)\big(-S_{\mathrm{M}}(\omega,\phi)+S_{\mathrm{M}}(\omega,\phi)\big)=0, \end{aligned}$$

we have $\pi_\omega(b_n - a_n) \to 0$ strongly because $\Omega_\omega$ is separating for $\pi_\omega(\mathcal{A})''$. Hence $\pi_\omega(a_n) \to -S_{\mathrm{M}}(\omega,\phi)\mathbf{1}$ strongly. □

**Lemma 4.4.** *Let $n(1) < n(2) < \cdots$ be positive integers, and let $a_k \in \mathcal{A}_{n(k)}$ be a positive contraction for each $k \in \mathbb{N}$.*

(i) *If $\inf_k \omega(a_k) > 0$, then*

$$\lim_{k\to\infty} \frac{1}{n(k)} \log \phi^G_{n(k)}(a_k) = 0.$$

(ii) *If $\inf_k \phi^G_{n(k)}(a_k) > 0$, then $\inf_k \omega(a_k) > 0$.*

(iii) *If $\lim_{k\to\infty} \omega(a_k) = 1$, then $\lim_{k\to\infty} \phi^G_{n(k)}(a_k) = 1$.*

*The above assertions* (i)–(iii) *hold also for $\mathcal{F}_{n(k)}$, $\hat\omega$ and $\hat\phi^G_{n(k)}$ instead of $\mathcal{A}_{n(k)}$, $\omega$ and $\phi^G_{n(k)}$, respectively.*

*Proof.* The last assertion is contained in [13, Lemma 3.2].

Let

$$F(s_1,s_2) := s_1 \log\frac{s_1}{s_2} + (1-s_1)\log\frac{1-s_1}{1-s_2}, \qquad 0 \le s_1, s_2 \le 1.$$

If the conclusion of (i) does not hold, then one may assume by taking a subsequence that $\phi^G_{n(k)}(a_k) \le e^{-n(k)\eta}$, $k \in \mathbb{N}$, for some $\eta > 0$. Using the monotonicity of relative entropy ([22, 5.12 (iii)]) applied to the map $\alpha : \mathbb{C}^2 \to \mathcal{A}_{n(k)}$, $\alpha(t_1,t_2) := t_1 a_k + t_2(\mathbf{1} - a_k)$, we have

$$\begin{aligned} S(\omega_{n(k)}, \phi^G_{n(k)}) &\ge S(\omega_{n(k)}\circ\alpha, \phi^G_{n(k)}\circ\alpha) = F(\omega_{n(k)}(a_k), \phi^G_{n(k)}(a_k)) \\ &\ge -\log 2 - \omega(a_k)\log\phi^G_{n(k)}(a_k) - (1-\omega(a_k))\log(1-\phi^G_{n(k)}(a_k)) \\ &\ge -\log 2 + n(k)\eta\omega(a_k) \end{aligned}$$

and hence

$$\liminf_{k\to\infty} \frac{1}{n(k)} S(\omega_{n(k)}, \phi^G_{n(k)}) \ge \eta \inf_k \omega(a_k) > 0.$$

This contradicts the equality

$$\lim_{n\to\infty} \frac{1}{n} S(\omega_n, \phi^G_n) = S_{\mathrm{M}}(\omega,\phi) + \omega(A_\Phi) + p(\Phi,\phi) = 0,$$

which is seen from $S(\omega_n, \phi^G_n) = S(\omega_n, \phi_n) + \omega(H_n) + \log\phi(e^{-H_n})$ and (4.6). Hence (i) follows.

Furthermore, thanks to the monotonicity of relative entropy as above and (4.5), we have

$$\begin{aligned} F(\phi^G_{n(k)}(a_k), \omega(a_k)) &\le S(\phi^G_{n(k)}, \omega_{n(k)}) \\ &\le S(\phi^G_{n(k)}, \phi^G_{n(k)}) + 2\|W_{n(k)}\| = 2\|W_{n(k)}\|. \end{aligned}$$

This shows the boundedness of $F(\phi^G_{n(k)}(a_k), \omega(a_k))$, from which (ii) and (iii) are easily verified. □

*Proof of* (4.1). For each $\delta > 0$ and $n \in \mathbb{N}$, let $p_n$ be the spectral projection of $-\frac{1}{n}\log D_n$ corresponding to the interval $(-S_{\mathrm{M}}(\omega,\phi) - \delta, -S_{\mathrm{M}}(\omega,\phi) + \delta)$. Then we have

$$\exp\bigl(n(-S_{\mathrm{M}}(\omega,\phi) - \delta)\bigr) D_n p_n \le p_n \le \exp\bigl(n(-S_{\mathrm{M}}(\omega,\phi) + \delta)\bigr) D_n p_n, \tag{4.8}$$

and Lemma 4.3 implies that $\pi_\omega(p_n) \to \mathbf{1}$ strongly as $n \to \infty$. Choose a sequence $n(1) < n(2) < \cdots$ such that

$$\lim_{k\to\infty} \frac{1}{n(k)} \log\beta_\varepsilon(\omega_{n(k)}, \phi_{n(k)}) = \liminf_{n\to\infty} \frac{1}{n}\log\beta_\varepsilon(\omega_n, \phi_n). \tag{4.9}$$

For each $k$ choose a projection $q_k \in \mathcal{A}_{n(k)}$ such that $\omega(q_k) \ge 1-\varepsilon$ and

$$\log \phi_{n(k)}(q_k) \le \log \beta_\varepsilon(\omega_{n(k)}, \phi_{n(k)}) + 1. \tag{4.10}$$

We may assume that $\pi_\omega(q_k)$ converges to some $y \in \pi_\omega(\mathcal{A})''$ weakly. Since $\pi_\omega(p_{n(k)}q_k) \to y$ weakly, we get

$$\lim_{k\to\infty} \omega(p_{n(k)}q_k) = \langle y\Omega_\omega, \Omega_\omega\rangle = \lim_{k\to\infty} \omega(q_k) \ge 1-\varepsilon$$

and by (4.8)

$$\phi(q_k) \ge \phi(p_{n(k)}q_k) \ge \exp\bigl(n(k)(-S_{\mathrm{M}}(\omega,\phi)-\delta)\bigr)\omega(p_{n(k)}q_k).$$

These give

$$\liminf_{k\to\infty} \frac{1}{n(k)} \log \phi(q_k) \ge -S_{\mathrm{M}}(\omega,\phi) - \delta. \tag{4.11}$$

Combining (4.9)–(4.11) yields

$$\liminf_{n\to\infty} \frac{1}{n} \log \beta_\varepsilon(\omega_n, \phi_n) \ge -S_{\mathrm{M}}(\omega,\phi) - \delta.$$

On the other hand, we obtain

$$\limsup_{n\to\infty} \frac{1}{n} \log \beta_\varepsilon(\omega_n, \phi_n) \le -S_{\mathrm{M}}(\omega,\phi) + \delta,$$

because by (4.8)

$$\begin{aligned} \frac{1}{n}\log \beta_\varepsilon(\omega_n,\phi_n) &\le \frac{1}{n}\log\phi(p_n) \le -S_{\mathrm{M}}(\omega,\phi) + \delta + \frac{1}{n}\log\omega(p_n) \\ &\le -S_{\mathrm{M}}(\omega,\phi) + \delta \end{aligned}$$

if $n$ is so large that $\omega(p_n) \ge 1-\varepsilon$. Thus, the proof of (4.1) is completed. □

*Proof of* (4.2). This can be proven by use of (i)–(iii) of Lemma 4.4 similarly to the proof of [13, Theorem 3.3]. Since the proof of the second inequality is a bit more involved than the first, we only prove the second.

Let $(\hat\pi, \hat{\mathcal{H}}, \hat\Omega)$ be the cyclic representation of $\mathcal{F}$ associated with $\hat\omega$. For each $\delta > 0$ and $n \in \mathbb{N}$, let $p_n$ be the spectral projection of $-\frac{1}{n}\log D_n^G$ to $(-S_{\mathrm{M}}(\omega,\phi)-\delta, S_{\mathrm{M}}(\omega,\phi)+\delta)$. Since $\frac{1}{n}\hat\pi(H_n) \to \hat\omega(A_\Phi)\mathbf{1} = \omega(A_\Phi)\mathbf{1}$ and hence $\frac{1}{n}\hat\pi(-\log D_n^G) \to -S_{\mathrm{M}}(\omega,\phi)\mathbf{1}$ strongly as (4.6) and (4.7), it follows that $\hat\pi(p_n) \to \mathbf{1}$ strongly as $n\to\infty$. Furthermore, we have

$$\exp\bigl(n(-S_{\mathrm{M}}(\omega,\phi)-\delta)\bigr)\frac{e^{-H_n}p_n}{\phi(e^{-H_n})} \le p_n \le \exp\bigl(n(-S_{\mathrm{M}}(\omega,\phi)+\delta)\bigr)\frac{e^{-H_n}p_n}{\phi(e^{-H_n})}. \tag{4.12}$$

Choose $n(1) < n(2) < \cdots$ such that

$$\lim_{k\to\infty} \frac{1}{n(k)} \log \beta_\varepsilon(\hat\phi^G_{n(k)}, \hat\phi_{n(k)}) = \liminf_{n\to\infty} \frac{1}{n}\log\beta_\varepsilon(\hat\phi^G_n, \hat\phi_n). \tag{4.13}$$

For each $k$ there is a projection $q_k \in \mathcal{F}_{n(k)}$ such that $\hat\phi^G_{n(k)}(q_k) \ge 1-\varepsilon$ and

$$\log \hat\phi_{n(k)}(q_k) \le \log\beta_\varepsilon(\hat\phi^G_{n(k)}, \hat\phi_{n(k)}) + 1. \tag{4.14}$$

Here, we may assume that $\hat\pi(q_k)$ converges to some $y \in \hat\pi(\mathcal{F})''$ weakly. Then we obtain

$$\lim_{k\to\infty} \hat\omega(p_{n(k)}q_kp_{n(k)}) = \langle y\hat\Omega, \hat\Omega\rangle = \lim_{k\to\infty}\hat\omega(q_k) > 0$$

by Lemma 4.4 (ii) (for $\hat\omega$ and $\hat\phi^G_{n(k)}$ with $a_k = q_k$), and hence

$$\lim_{k\to\infty} \frac{1}{n(k)} \log \hat\phi^G_{n(k)}(p_{n(k)} q_k p_{n(k)}) = 0 \tag{4.15}$$

by Lemma 4.4 (i) (for $\hat\omega$ and $\hat\phi^G_{n(k)}$ with $a_k = p_{n(k)} q_k p_{n(k)}$). Furthermore, since $p_n$ commutes with $e^{-H_n}$ and $\bigotimes_1^n e^{-h}$, we obtain

$$\begin{aligned}
\hat\phi_{n(k)}(q_k) &= d^{-n(k)} \mathrm{Tr}_{\mathcal{F}_{n(k)}}\left(\left(\bigotimes_1^{n(k)} e^{-h}\right) q_k\right) \\
&\ge d^{-n(k)} \mathrm{Tr}_{\mathcal{F}_{n(k)}}\left(\left(\bigotimes_1^{n(k)} e^{-h}\right) p_{n(k)} q_k\right) \\
&\ge \exp\bigl(n(k)(-S_{\mathrm{M}}(\omega,\phi)-\delta)\bigr) \frac{d^{-n(k)} \mathrm{Tr}_{\mathcal{F}_{n(k)}}\bigl(\bigl(\bigotimes_1^{n(k)} e^{-h}\bigr) e^{-H_{n(k)}} p_{n(k)} q_k\bigr)}{\phi(e^{-H_{n(k)}})} \\
&= \exp\bigl(n(k)(-S_{\mathrm{M}}(\omega,\phi)-\delta)\bigr) \frac{\hat\phi(e^{-H_{n(k)}} p_{n(k)} q_k p_{n(k)})}{\phi(e^{-H_{n(k)}})} \\
&= \exp\bigl(n(k)(-S_{\mathrm{M}}(\omega,\phi)-\delta)\bigr) \hat\phi^G_{n(k)}(p_{n(k)} q_k p_{n(k)})
\end{aligned}$$

using (4.12) and (3.1). This together with (4.13)–(4.15) yields

$$\liminf_{n\to\infty} \frac{1}{n} \log \beta_\varepsilon(\hat\phi^G_n, \hat\phi_n) \ge -S_{\mathrm{M}}(\omega,\phi) - \delta.$$

On the other hand, since $\hat\phi^G_n(p_n) \to 1$ by Lemma 4.4 (iii) (for $\hat\omega$ and $\hat\phi^G_n$), we have $\hat\phi^G_n(p_n) \ge 1-\varepsilon$ for large $n$, and for such $n$

$$\frac{1}{n} \log \beta_\varepsilon(\hat\phi^G_n, \hat\phi_n) \le \frac{1}{n} \log \hat\phi_n(p_n) \le -S_{\mathrm{M}}(\omega,\phi) + \delta$$

thanks to (4.12). Therefore,

$$\limsup_{n\to\infty} \frac{1}{n} \log \beta_\varepsilon(\hat\phi^G_n, \hat\phi) \le -S_{\mathrm{M}}(\omega,\phi) + \delta,$$

completing the proof of (4.2). □

*Proof of* (4.3) *and* (4.4). Assume that $\xi$ is central. Since $s_{\mathcal{A}}(\omega) = s_{\mathcal{F}}(\hat\omega)$ by Theorem 3.1, the assertion (4.4) is contained in [13, Theorem 3.3]. To prove (4.3), we first assume that $\xi$ is trivial. Then, by Lemma 2.1 (1) and (4.1) (in case of $\phi = \nu$) we have

$$\begin{aligned}
s_{\mathcal{A}}(\omega) &= -S_{\mathrm{M}}(\omega,\nu) + \log d \\
&= \lim_{n\to\infty} \frac{1}{n} \log \beta_\varepsilon(\omega_n, \nu_n) + \log d \\
&= \lim_{n\to\infty} \frac{1}{n} \log \beta_\varepsilon(\omega_n).
\end{aligned}$$

The latter equality in the above is readily verified from (2.4) and (2.5). The other equality in (4.3) when $\phi = \nu$ is similarly shown from the first equality in (4.2). When $\xi$ is not trivial, we consider $\Phi^h$ belonging to $\mathcal{B}_0(\mathcal{A})$ instead of $\Phi$. Note that $\omega$ is an $\alpha^{\Phi^h}$-KMS state with trivial chemical potential and $\phi^G_n$ is the local Gibbs state with respect to $\Phi^h$ and $\nu$. Hence, the above special case gives the conclusion. □

## 5. Remarks and problems

Some problems as well as related known results are in order.

**5.1.** It is known [11, 23] that the weak*-closure of $\mathcal{ET}^{\mathrm{f}}(\mathcal{A})$ coincides with the set $\mathcal{ET}(\mathcal{A})$ of all extremal tracial states of $\mathcal{A}$ as far as $G$ is a compact connected Lie group. For $\Phi \in \mathcal{B}_0(\mathcal{A})$ let $\mathcal{E}K(\mathcal{A},\Phi)$ denote the set of all extremal $\alpha^\Phi$-KMS states of $\mathcal{A}$ (see Proposition 1.1) and $\mathcal{E}K^{\mathrm{f}}(\mathcal{A},\Phi)$ the set of all faithful $\omega \in \mathcal{E}K(\mathcal{A},\Phi)$. Theorems 1.5 and 1.6 say that there is a bijective correspondence $\phi \leftrightarrow \omega$ between $\mathcal{ET}^{\mathrm{f}}(\mathcal{A})$ and $\mathcal{E}K^{\mathrm{f}}(\mathcal{A},\Phi)$. We further know (see [14, Theorem 4.6]) that the correspondence $\phi \mapsto \omega$ is a weak*-homeomorphism from $\mathcal{ET}^{\mathrm{f}}(\mathcal{A})$ onto $\mathcal{E}K^{\mathrm{f}}(\mathcal{A},\Phi)$. Upon these considerations we are interested in the following problems:

(1) Does the weak*-closure of $\mathcal{E}K^{\mathrm{f}}(\mathcal{A},\Phi)$ coincide with $\mathcal{E}K(\mathcal{A},\Phi)$ (as far as $G$ is a compact connected Lie group)?
(2) Does the above $\phi \mapsto \omega$ extend to a weak*-homeomorphism from $\mathcal{ET}(\mathcal{A})$ onto $\mathcal{E}K(\mathcal{A},\Phi)$?

**5.2.** In the situation of Theorem 3.1 it seems that the equality $s_{\mathcal{A}}(\omega) = s_{\mathcal{F}}(\hat\omega)$ holds without the assumption of $\xi$ being central. This is equivalent to the equality $s_{\mathcal{A}}(\omega) = \lim_{n\to\infty} \frac{1}{n} S(\phi_n^G)$, which is the only missing point in Theorem 3.1.

**5.3.** The equality $-S_{\mathrm{M}}(\omega,\phi) = \lim_{n\to\infty} \frac{1}{n}\beta_\varepsilon(\hat\omega_n,\hat\phi_n)$ is missing in Theorem 4.1, which is equivalent to

$$-S_{\mathrm{M}}(\hat\omega,\hat\phi) = \lim_{n\to\infty} \frac{1}{n}\log\beta_\varepsilon(\hat\omega_n,\hat\phi_n) \tag{5.1}$$

due to Theorem 3.1. Note that $\hat\phi$ is a product state of $\mathcal{F}$ and $\hat\omega$ is completely ergodic, i.e., extremal for all $\theta^n$, $n \ge 1$. Thus, the equality (5.1) is an old open problem from the viewpoint of quantum hypothesis testing in [12], where the weaker result was proven:

$$\begin{aligned} -S_{\mathrm{M}}(\hat\omega,\hat\phi) &\ge \limsup_{n\to\infty} \frac{1}{n}\log\beta_\varepsilon(\hat\omega_n,\hat\phi_n), \\ -\frac{1}{1-\varepsilon}S_{\mathrm{M}}(\hat\omega,\hat\phi) &\le \liminf_{n\to\infty} \frac{1}{n}\log\beta_\varepsilon(\hat\omega_n,\hat\phi_n). \end{aligned}$$

In this connection, it is worthwhile to note that T. Ogawa and H. Nagaoka established in [21] the equality

$$-S(\varphi,\psi) = \lim_{n\to\infty} \frac{1}{n}\log\beta_\varepsilon(\varphi_n,\psi_n)$$

when $\varphi,\psi$ are states of $M_d(\mathbb{C})$ and $\varphi_n,\psi_n$ are the $n$-fold tensor products of $\varphi,\psi$. The problem of macroscopic uniformity for states of spin $C^*$-algebras was completely solved in a recent paper by I. Bjelaković et al. as follows: If $\varphi$ is an extremal translation-invariant state of the $\nu$-dimensional spin algebra $\bigotimes_{\mathbb{Z}^\nu} M_d(\mathbb{C})$, then

$$s(\varphi) = \lim_{\Lambda\to\mathbb{Z}^\nu} \frac{1}{|\Lambda|}\log\beta_\varepsilon(\varphi)$$

for any $0 < \varepsilon < 1$. See [6] for details.

**5.4.** Although many arguments in this paper as well as in [14] work also in gauge-invariant $C^*$-systems over the multi-dimensional lattice $\mathbb{Z}^\nu$, some difficulties arise when we would extend our whole arguments to the multi-dimensional case. For instance, it does not seem that Proposition 1.1 holds in multi-dimensional gauge-invariant $C^*$-systems. The proposition is crucial when we use the chemical potential theory as in the proof of Theorem 1.6. Moreover, the assumption of uniformly bounded surface energies is sometimes useful in our discussions. In the multi-dimensional case, the assumption is obviously too strong and, if it is not assumed, the non-uniqueness of KMS states (or the phase transition) can occur. Indeed, the

uniqueness of $\alpha^{\Phi^h}$-KMS state of $\mathcal{F}$ is essential in the proof of Theorem 1.5. Consequently, some new ideas must be needed to extend the theory to the multi-dimensional setting.

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